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Jack polynomials with prescribed symmetry and hole propagator of spin Calogero–Sutherland model

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Abstract. We study the hole propagator of the Calogero–Sutherland model with SU(2) internal symmetry. We obtain the exact expression for arbitrary non-negative integer coupling parameter β and prove the conjecture proposed by one of the authors. Our method is based on the theory of the Jack polynomials with a prescribed symmetry.

1. Introduction

One of the goals in physics of interacting particle systems is to understand the dynamics. In particular, dynamical properties in one-dimensional systems are anticipated to be intriguing, because interaction effects are crucial and naive pictures based on perturbations lose their validity. As a non-perturbative approach, conformal field theory is a powerful method with which to study the low-energy physics of the Tomonaga–Luttinger liquid. Beyond the conformal limit, on the other hand, integrable systems give us opportunities for analytical study of dynamics. Among them, the Calogero–Sutherland (CS) model [1, 2] of particles interacting with the two-body inverse square interaction provides the simplest example of systems with non-trivial dynamics. For the spinless CS model, the density–density correlation function [3–6], hole propagator [4, 5, 7] and particle propagator [8, 9] have been obtained analytically.

The spinless CS model has a number of variants. One of them is the spin CS model [10, 11]. This model describes n particles with coordinates $X = (X_1, \ldots, X_n)$ moving along a circle of length L and with a spin with p possible values. The Hamiltonian of the model is given by

$$\hat{H}_n = -\sum_{i=1}^{n} \frac{\partial^2}{\partial X_i^2} + 2\left(\frac{\pi}{L}\right)^2 \sum_{1\leqslant i<j\leqslant n} \frac{\beta(\beta + P_{ij})}{\sin^2 \frac{\pi}{L}(X_i - X_j)} \tag{1}$$

where β is a coupling parameter and P_{ij} is the spin-exchange operator. In this paper, we take β to be a non-negative integer.

The spin CS model with $p = 2$ is particularly relevant to condensed matter physics; most one-dimensional systems are realized experimentally in electron systems and hence we should take account of spin degrees of freedom of each particle. Furthermore, we can regard the spin CS model with $p = 2$ as a one-dimensional variant of a singlet fractional quantum Hall (FQH) system; the ground state of (1) can be derived from the Halperin

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wavefunction [12] for the singlet FQH state by the restriction of particle coordinates on a ring in a two-dimensional plane.

For the dynamics of the spin CS model with $p = 2$, the expression for the hole part of Green function has been proposed [13] relying on finite-size calculations. Subsequently the dynamical density–density and spin–spin correlation functions have been derived by Uglov [14] in an exact treatment. Since [14] appeared, however, exact derivations of the hole propagator have still been missing. Recently, in [15], Dunkl has developed the theory of the Jack polynomials with a prescribed symmetry; those polynomials are symmetric or alternating with respect to the interchange of certain subsets of variables. Dunkl's results allow us to derive 'the binomial formula', which is directly related to the matrix element of the local field operator in the spin CS model.

The aim of this paper is to prove the earlier conjecture on the hole propagator [13] utilizing Dunkl's results [15]. Though the method in this paper is also applicable to the general SU(p) case, we concentrate on the SU(2) case for simplicity.

Here we recall the conjecture in [13] on the hole propagator. In the thermodynamic limit, the expression for the hole propagator $G(r, t)$ is expected to be

$$G(r,t) = c(\beta)\prod_{k=1}^{\beta}\int_{-1}^{1}\mathrm{d}u_k \prod_{l=1}^{\beta+1}\int_{-1}^{1}\mathrm{d}v_l |F(u,v)|^2 \exp[-\mathrm{i}(E(u,v)t - Q(u,v)r)]. \tag{2}$$

Here $c(\beta)$ is a constant factor and $u = (u_1, \ldots, u_\beta)$, $v = (v_1, \ldots, v_{\beta+1})$ represent normalized velocities of the quasiholes. In the expression (2), F, Q and E represent the form factor, momentum and energy, respectively. The explicit forms of them are as follows. The form factor F is given by

$$F(u,v) = \frac{\prod_{1\leqslant k<l\leqslant\beta}(u_k-u_l)^{g_\mathrm{d}}\prod_{1\leqslant k<l\leqslant\beta+1}(v_k-v_l)^{g_\mathrm{d}}\prod_{k=1}^{\beta}\prod_{l=1}^{\beta+1}(u_k-v_l)^{g_\mathrm{o}}}{\prod_{k=1}^{\beta}(1-u_k^2)^{(1-g_\mathrm{d})/2}\prod_{l=1}^{\beta+1}(1-v_l^2)^{(1-g_\mathrm{d})/2}} \tag{3}$$

where $g_\mathrm{d} = (\beta+1)/(2\beta+1)$ and $g_\mathrm{o} = -\beta/(2\beta+1)$. The momentum Q and energy E are given, respectively, by

$$Q(u,v) = \frac{\pi\rho_0}{2}\left(\sum_{k=1}^{\beta}u_k + \sum_{l=1}^{\beta+1}v_l\right) \tag{4}$$

$$E(u,v) = -(2\beta+1)(\frac{\pi\rho_0}{2})^2\left(\sum_{k=1}^{\beta}u_k^2 + \sum_{l=1}^{\beta+1}v_l^2\right). \tag{5}$$

Here $\rho_0 = 2M/L$ is the mean density of particles. In what follows, we prove this conjecture and show that the constant $c(\beta)$ is given by

$$c(\beta) = \frac{\rho_0}{4(2\beta+1)^\beta\Gamma(\beta+2)}\prod_{k=1}^{2\beta+1}\frac{\Gamma((\beta+1)/(2\beta+1))}{\Gamma(k/(2\beta+1))^2} \tag{6}$$

where $\Gamma(z)$ is the gamma function. The physical implication of expressions (2)–(4) has been discussed in [13, 16].

This paper is organized as follows. In section 2, we define the Jack polynomials with a prescribed symmetry and discuss their basic properties. In particular, we derive the binomial formula of the Jack polynomials with the prescribed symmetry using Dunkl's results. In section 3, we obtain the hole propagator for a finite number of particles using the mathematical formulae presented in section 2 and present the derivation of expression (2).

2. Jack polynomials with prescribed symmetry

In this section, we present mathematical results necessary to derive the expression for the hole propagator. First, as a preliminary, we fix our notations. Second, we define the non-symmetric Jack polynomials. Third, we define the Jack polynomials with a prescribed symmetry in terms of the non-symmetric Jack polynomials. Fourth, we present the basic formulae of the Jack polynomials with the prescribed symmetry: the norm, Cauchy formula, and evaluation formula are discussed. Last, from the Cauchy and evaluation formulae, we derive the binomial formula, which gives the matrix element of the local field operator in the calculation of the hole propagator.

2.1. Notations

First of all, we fix notations (see [17–19]). For a fixed non-negative integer n, let $\Lambda_n = \{\eta = (\eta_1, \eta_2, \ldots, \eta_n) | \eta_i \in \mathbb{Z}_{\geqslant 0}, 1 \leqslant i \leqslant n\}$ be the set of all compositions with length less than or equal to n. The weight $|\eta|$ of a composition $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \Lambda_n$ is defined by $|\eta| = \sum_{i=1}^{n} \eta_i$. The length $l(\eta)$ of η is defined as the number of non-zero η_i in η. The set of all partitions with length less than or equal to n is defined by $\Lambda_n^+ = \{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda_n | \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant 0\}$. The dominance order $<$ on partitions is defined as follows: for $\lambda, \mu \in \Lambda_n^+$, $\lambda \leqslant \mu$ if $|\lambda| = |\mu|$ and $\sum_{i=1}^{k} \lambda_i \leqslant \sum_{i=1}^{k} \mu_i$ for all $k = 1, \ldots, n$. For a composition $\eta \in \Lambda_n$, we denote by η^+ the (unique) partition which is a rearrangement of the composition η. Now we define a partial order $\prec$ on compositions as follows: for $\nu, \eta \in \Lambda_n$, $\nu \prec \eta$ if $\nu^+ < \eta^+$ with dominance ordering on partitions or if $\nu^+ = \eta^+$ and $\sum_{i=1}^{k} \nu_i \leqslant \sum_{i=1}^{k} \eta_i$ for all $k = 1, \ldots, n$.

For a given composition $\eta = (\eta_1, \eta_2, \ldots, \eta_n) \in \Lambda_n$ and pairs of integers $s = (i, j)$ satisfying $1 \leqslant i \leqslant l(\eta)$ and $1 \leqslant j \leqslant \eta_i$, we define the following quantities:

$$a(s) = \eta_i - j \tag{7}$$

$$a'(s) = j - 1 \tag{8}$$

$$l(s) = \#\{k \in \{1, \ldots, i-1\} | j \leqslant \eta_k + 1 \leqslant \eta_i\} + \#\{k \in \{i+1, \ldots, n\} | j \leqslant \eta_k \leqslant \eta_i\} \tag{9}$$

$$l'(s) = \#\{k \in \{1, \ldots, i-1\} | \eta_k \geqslant \eta_i\} + \#\{k \in \{i+1, \ldots, n\} | \eta_k > \eta_i\}. \tag{10}$$

Here, for a set A, $\#A$ denotes the number of elements. In the above expressions, $a(s)$, $a'(s)$, $l(s)$ and $l'(s)$ are called arm-, coarm-, leg-, and coleg-lengths, respectively. Furthermore, for a composition $\eta \in \Lambda_n$ and a parameter β, we define the following four quantities:

$$d_\eta = \prod_{s \in \eta} ((a(s) + 1)/\beta + l(s) + 1) \tag{11}$$

$$d'_\eta = \prod_{s \in \eta} ((a(s) + 1)/\beta + l(s)) \tag{12}$$

$$e_\eta = \prod_{s \in \eta} ((a'(s) + 1)/\beta + n - l'(s)) \tag{13}$$

$$e'_\eta = \prod_{s \in \eta} ((a'(s) + 1)/\beta + n - l'(s) - 1). \tag{14}$$

2.2. Non-symmetric Jack polynomials

Now we define the non-symmetric Jack polynomials [20, 21]. For this purpose, we define the Cherednik–Dunkl operators [22, 23] as

$$\hat{d}_i = x_i \frac{\partial}{\partial x_i} + \beta \sum_{k=1}^{i-1} \frac{x_i}{x_i - x_k}(1 - s_{ik}) + \beta \sum_{k=i+1}^{n} \frac{x_k}{x_i - x_k}(1 - s_{ik}) + \beta(1 - i) \tag{15}$$

for $1 \leqslant i \leqslant n$. The operator $s_{ij} = (i, j)$ is the transposition which interchanges coordinates x_i and x_j (s_{ij} is called the coordinate exchange operator [24]). The operator $\hat{d}_i$ is a mapping in homogeneous polynomials of $x = (x_1, \ldots, x_n)$. Furthermore, all the operators $\{\hat{d}_i\}$ commute with each other, and hence these operators can be diagonalized simultaneously.

For a given composition η, the non-symmetric Jack polynomial $E_\eta(x; 1/\beta)$ is defined as the $|\eta|$th order homogeneous polynomial satisfying the following two conditions.

(1) The polynomial $E_\eta(x; 1/\beta)$ has the form of

$$E_\eta(x; 1/\beta) = x^\eta + \sum_{\substack{\nu \in \Lambda_n \\ \nu \prec \eta}} c_{\nu\eta} x^\nu \tag{16}$$

in terms of the monomials $x^\nu = x_1^{\nu_1}, \ldots, x_n^{\nu_n}$.

(2) $E_\eta(x; 1/\beta)$ is a simultaneous eigenfunction of $\hat{d}_i$ for $1 \leqslant i \leqslant n$.

Here we remark on two important properties of the non-symmetric Jack polynomial E_η. One is the eigenvalue of E_η for $\hat{d}_i$, which is given by

$$\bar{\eta}_i = \eta_i - \beta(\#\{k \in \{1, \ldots, i-1\} | \eta_k \geqslant \eta_i\} + \#\{k \in \{i+1, \ldots, n\} | \eta_k > \eta_i\}). \tag{17}$$

The other is the action of the transposition $s_i := s_{i,i+1} = (i, i+1)$ on E_η [18, 19]

$$s_i E_\eta = \begin{cases} \xi_i E_\eta + (1 - \xi_i^2) E_{s_i\eta} & \eta_i > \eta_{i+1} \\ E_\eta & \eta_i = \eta_{i+1} \\ \xi_i E_\eta + E_{s_i\eta} & \eta_i < \eta_{i+1} \end{cases} \tag{18}$$

where $\xi_i = \beta/(\bar{\eta}_i - \bar{\eta}_{i+1})$. In particular, property (18) plays an important role in the proof of the basic properties of the Jack polynomials with a prescribed symmetry.

2.3. Jack polynomials with prescribed symmetry

Next we define the Jack polynomials with a prescribed symmetry. For this purpose, we introduce some notations. The interval $I = [1, n]$ denotes $\{i \in \mathbb{Z} | 1 \leqslant i \leqslant n\}$ for a positive integer n. For an integer $m \in I$, we define $I_\downarrow = [1, m]$ and $I_\uparrow = [m+1, n]$. In addition, we introduce the notations $n_\downarrow = \#I_\downarrow (= m)$ and $n_\uparrow = \#I_\uparrow (= n - m)$. We consider the subgroup $S_{n_\downarrow} \times S_{n_\uparrow}$ of the symmetric group S_n which leaves $\{1, \ldots, n_\downarrow\}$ and $\{n_\downarrow + 1, \ldots, n\}$ invariant. Further we define $\Lambda_n^{\downarrow\uparrow} \subset \Lambda_n$ as a set of 'partial partitions'

$$\begin{aligned} \Lambda_n^{\downarrow\uparrow} = \{\mu = (\mu^\downarrow, \mu^\uparrow) \\ = (\mu_1^\downarrow, \ldots, \mu_{n_\downarrow}^\downarrow, \mu_1^\uparrow, \ldots, \mu_{n_\uparrow}^\uparrow) \in \Lambda_n | \mu_1^\downarrow > \cdots > \mu_{n_\downarrow}^\downarrow, \mu_1^\uparrow > \cdots > \mu_{n_\uparrow}^\uparrow\}. \end{aligned} \tag{19}$$

Now we consider polynomials which are alternating under the action of $S_{n_\downarrow} \times S_{n_\uparrow}$ [15, 25]. We define the (alternating) Jack polynomial with the prescribed symmetry $K_\mu(x; \beta)$ for $\mu = (\mu^\downarrow, \mu^\uparrow) \in \Lambda_n^{\downarrow\uparrow}$ by the following two conditions.

(1) The polynomial K_μ has the form of

$$K_\mu(x; \beta) = \sum_{\eta = (\eta^\downarrow, \eta^\uparrow)} a_\eta E_\eta(x; 1/\beta) \tag{20}$$

with the normalization $a_\mu = 1$. Here the sums over $\eta^\downarrow$ and $\eta^\uparrow$ are taken on the whole rearrangements of $\mu^\downarrow$ and $\mu^\uparrow$, respectively.

(2) Under the action of the transposition s_i, the polynomial $K_\mu(x)$ is transformed as $s_i K_\mu(x) = -K_\mu(x)$ for $i \in I_\downarrow \setminus \{n_\downarrow\}$ or $\in I_\uparrow \setminus \{n\}$. (Here, for a set A and its subset B, we denote by $A \backslash B$ the complementary set of B in A.)

From the above definition and (18), we can derive the recursion relation for the coefficient a_η

$$a_{s_i\eta} = -\frac{\bar{\eta}_i - \bar{\eta}_{i+1} - \beta}{\bar{\eta}_i - \bar{\eta}_{i+1}} a_\eta \tag{21}$$

where $i \in I_\downarrow \setminus \{n_\downarrow\}$ or $\in I_\uparrow \setminus \{n\}$. From the alternating property of K_μ, we can also write as

$$\rho_\mu K_\mu(x) = \sum_{\sigma \in S_{n_\downarrow} \times S_{n_\uparrow}} \mathrm{sgn}(\sigma) \sigma E_\mu(x) \tag{22}$$

where the symbol $\mathrm{sgn}\,(\sigma)$ denotes the sign of the permutation σ. The expression for the factor ρ_μ can be obtained from the normalization $a_\mu = 1$ and relation (18) as

$$\rho_\mu = \prod_{s=\downarrow,\uparrow} \prod_{\substack{i,j \in I_s \\ i<j}} \frac{\bar{\mu}_i - \bar{\mu}_j - \beta}{\bar{\mu}_i - \bar{\mu}_j} \tag{23}$$

for $\mu \in \Lambda_n^{\downarrow\uparrow}$.

2.4. Basic properties

Next we discuss the basic properties of the Jack polynomials with the prescribed symmetry. The combinatorial norm, integral norm, Cauchy formula and evaluation formula are discussed.

(a) Combinatorial norm: define the polynomials $\{q_\eta(x)\}_{\eta \in \Lambda_n}$ by

$$\Omega(x|y) = \prod_{i=1}^{n} (1 - x_i y_i)^{-1} \prod_{i,j=1}^{n} (1 - x_i y_j)^{-\beta} = \sum_{\eta \in \Lambda_n} q_\eta(x) y^\eta. \tag{24}$$

The (combinatorial) inner product $\langle \bullet, \bullet \rangle_n^{\mathrm{c}}$ is then defined by $\langle q_\nu, x^\eta \rangle_n^{\mathrm{c}} = \delta_{\nu\eta}$ [26]. The Jack polynomials with the prescribed symmetry are orthogonal with respect to the inner product $\langle \bullet, \bullet \rangle_n^{\mathrm{c}}$ [25]. Using the norm formula $(||E_\eta||_n^{\mathrm{c}})^2 = \langle E_\eta, E_\eta \rangle_n^{\mathrm{c}} = d'_\eta / d_\eta$ for $\eta \in \Lambda_n$ [19] and the transformation properties (18) of the non-symmetric Jack polynomials, we can prove

$$(||K_\mu||_n^{\mathrm{c}})^2 = \langle K_\mu, K_\mu \rangle_n^{\mathrm{c}} = \frac{n_\downarrow! n_\uparrow!}{\rho_\mu} \frac{d'_\mu}{d_\mu} \tag{25}$$

for $\mu \in \Lambda_n^{\downarrow\uparrow}$.

(b) Integral norm: for functions $f(x)$ and $g(x)$ in complex variables $x = (x_1, \ldots, x_n)$, we define the inner product $\langle \bullet, \bullet \rangle_n^0$ by the following formula:

$$\langle f, g \rangle_n^0 = \prod_{i=1}^{n} \oint_{|x_i|=1} \frac{\mathrm{d}x_i}{2\pi \mathrm{i} x_i} f(x) \overline{g(x)} |\Delta(x)|^{2\beta}. \tag{26}$$

Here $\Delta(x) = \prod_{\substack{i,j \in I \\ i<j}} (x_i - x_j)$ is the VanderMonde determinant and $\overline{g(x)}$ denotes the complex conjugation of $g(x)$. The Jack polynomials with the prescribed symmetry are orthogonal with respect to the inner product $\langle \bullet, \bullet \rangle_n^0$ [25]. It is known [27] that

$$\langle E_\eta, E_\eta \rangle_n^0 / \langle E_\eta, E_\eta \rangle_n^{\mathrm{c}} = \frac{\Gamma(n\beta + 1)}{\Gamma(\beta + 1)^n} \frac{e_\eta}{e'_\eta} \tag{27}$$

for $\eta \in \Lambda_n$. We note that the right-hand side of (27) depends on η through η^+. Thus relation (27) immediately leads to

$$\langle K_\mu, K_\mu\rangle_n^0 / \langle K_\mu, K_\mu\rangle_n^{\mathrm{c}} = \frac{\Gamma(n\beta+1)}{\Gamma(\beta+1)^n}\frac{e_\mu}{e'_\mu}. \tag{28}$$

As a result of (25) and (28), we obtain

$$(||K_\mu||_n^0)^2 = \langle K_\mu, K_\mu\rangle_n^0 = \frac{n_\downarrow! n_\uparrow!}{\rho_\mu}\frac{\Gamma(n\beta+1)}{\Gamma(\beta+1)^n}\frac{e_\mu d'_\mu}{e'_\mu d_\mu} \tag{29}$$

for $\mu \in \Lambda_n^{\downarrow\uparrow}$.

(c) The Cauchy formula: for the coordinate $x = (x_1, \ldots, x_n)$ and a fixed integer $m \in I$, we define $x^\downarrow = (x_1, \ldots, x_m)$ and $x^\uparrow = (x_{m+1}, \ldots, x_n)$. The Cauchy formula for the Jack polynomials with the prescribed symmetry is given by

$$\prod_{s=\downarrow,\uparrow}\prod_{i,j\in I_s}(1-x_iy_j)^{-1}\prod_{i,j\in I}(1-x_iy_j)^{-\beta} = n_\downarrow! n_\uparrow! \sum_{\mu\in\Lambda_n^{\downarrow\uparrow}}(||K_\mu||_n^{\mathrm{c}})^{-2}\tilde{K}_\mu(x)\tilde{K}_\mu(y) \tag{30}$$

where $\tilde{K}_\mu(x) = K_\mu(x)/(\Delta(x^\downarrow)\Delta(x^\uparrow))$ with $\Delta(x^s) = \prod_{\substack{i,j\in I_s\\ i<j}}(x_i - x_j)$, $(s = \downarrow, \uparrow)$. The proof of the Cauchy formula (30) is based on Cauchy's formula for the non-symmetric polynomials $\Omega(x|y) = \sum_{\eta\in\Lambda_n}(||E_\eta||_n^{\mathrm{c}})^{-2}E_\eta(x)E_\eta(y)$ [19] and Cauchy's determinant identity (see also [27]). The proof also requires transformation properties (18) and (21) together with those for d_η and d'_η [19].

(d) evaluation formula: using the evaluation formula for the non-symmetric Jack polynomials $E_\eta(\underbrace{1,\ldots,1}_{n}) = e_\eta/d_\eta$ [19] and new skew operators [15], Dunkl obtained the evaluation formula for the Jack polynomials with the prescribed symmetry†

$$\tilde{K}_\mu(\underbrace{1,\ldots,1}_{n}) = \beta^{-|\delta|}\frac{1}{e_\delta}\frac{e_\mu}{d_\mu}\frac{\pi_\mu}{\rho_\mu} \tag{31}$$

where

$$\pi_\mu = \prod_{s=\downarrow,\uparrow}\prod_{\substack{i,j\in I_s\\ i<j}}(\bar{\mu}_i - \bar{\mu}_j - \beta) \tag{32}$$

for $\mu \in \Lambda_n^{\downarrow\uparrow}$. In (31), the composition $\delta \in \Lambda_n^{\downarrow\uparrow}$ is introduced as $\delta = \delta(n^\downarrow, n^\uparrow) := (\delta^\downarrow, \delta^\uparrow)$ with $\delta^\downarrow = (n^\downarrow - 1, \ldots, 1, 0)$ and $\delta^\uparrow = (n^\uparrow - 1, \ldots, 1, 0)$.

2.5. Binomial formula

In this section, we derive the binomial formula with the use of the Cauchy formula (30) and evaluation formula (31). For $a \in \mathbb{C}$, the binomial formula for the Jack polynomials with the prescribed symmetry is given by

$$\prod_{s=\downarrow,\uparrow}\prod_{i\in I_s}(1-x_i)^{a-n_s} = \sum_{\mu\in\Lambda_n^{\downarrow\uparrow}}\chi_\mu(a)\tilde{K}_\mu(x) \tag{33}$$

where

$$\chi_\mu(a) = \beta^{-|\mu|}\frac{(1-a)_{\mu^+}}{(1-a)_{\delta^+}}\frac{\pi_\mu}{d'_\mu}. \tag{34}$$

† The polynomial E_η for a composition η in this paper is different from ζ_η in [15] by the constant factor d'_η/d_η; $E_\eta = (d'_\eta/d_\eta)\zeta_\eta$.

Here, for an indeterminate t, a partition λ and a parameter β, the generalized shifted factorial is defined by

$$(t)_\lambda = \prod_{i=1}^{n} \frac{\Gamma(t-\beta(i-1)+\lambda_i)}{\Gamma(1-\beta(i-1))}. \tag{35}$$

We can prove the formula (33) in a way similar to that used in lemma 5.2 in [19] and proposition 2.4 in [27].

Let n' be $n'_\downarrow + n'_\uparrow (\geqslant n)$ for that pair $(n'_\downarrow, n'_\uparrow)$ of positive integers $n'_\downarrow$ and $n'_\uparrow$ which satisfies $n'_\downarrow - n'_\uparrow = n_\downarrow - n_\uparrow$. We consider the Cauchy formula (30) in n' variables. First we discuss the left-hand side; the left-hand side of (30) becomes

$$\prod_{s=\downarrow,\uparrow} \prod_{i,j\in I'_s} (1-x_i y_j)^{-1} \prod_{i,j\in I'} (1-x_i y_j)^{-\beta} \tag{36}$$

with $I'_\downarrow = [1, n'_\downarrow]$, $I'_\uparrow = [n'_\downarrow + 1, n']$ and $I' = [1, n']$. Now we set $x_{n_\downarrow+1} = \cdots = x_{n'_\downarrow} = 0$, $x_{n'_\downarrow+n_\uparrow+1} = \cdots = x_{n'_\downarrow+n'_\uparrow} = 0$ and $y_1 = \cdots = y_{n'_\downarrow+n'_\uparrow} = 1$. Further we replace $(x_{n'_\downarrow+1}, \ldots, x_{n'_\downarrow+n_\uparrow})$ by $(x_{n_\downarrow+1}, \ldots, x_{n_\downarrow+n_\uparrow})$. Expression (36) then turns into

$$\prod_{i\in I_\downarrow=[1,n_\downarrow]} (1-x_i)^{-n'\beta-n'_\downarrow} \prod_{j\in I_\uparrow=[n_\downarrow+1,n]} (1-x_j)^{-n'\beta-n'_\downarrow+n_\downarrow-n_\uparrow}. \tag{37}$$

Next we discuss the right-hand side of the Cauchy formula in n' variables. We set $y_i = 1$ for $1 \leqslant i \leqslant n'$. With the use of evaluation formula (31), we immediately see that the right-hand side of the Cauchy formula becomes

$$\sum_\nu \beta^{-|\delta'|} \frac{e_\nu \pi_\nu}{e_{\delta'} d'_\nu} \tilde{K}_\nu(x). \tag{38}$$

Here the sum is taken over $\nu \in \Lambda_{n'}^{\downarrow\uparrow}$. The symbol $\delta' = (\delta'^\downarrow, \delta'^\uparrow)$ denotes the composition $\delta(n'_\downarrow, n'_\uparrow) \in \Lambda_{n'}^{\downarrow\uparrow}$.

Now we set $x_{n_\downarrow+1} = \cdots = x_{n'_\downarrow} = 0$ and $x_{n'_\downarrow+n_\uparrow+1} = \cdots = x_{n'_\downarrow+n'_\uparrow} = 0$ in (38). Non-vanishing contributions the sum in (38) then only come from the compositions $\nu = (\nu^\downarrow, \nu^\uparrow) \in \Lambda_{n'}^{\downarrow\uparrow}$ satisfying $l(\nu^\downarrow - \delta'^\downarrow) \leqslant n_\downarrow$ and $l(\nu^\uparrow - \delta'^\uparrow) \leqslant n_\uparrow$ where $\nu^s - \delta'^s = (\nu_1^s - \delta_1'^s, \ldots, \nu_{n'_s}^s - \delta_{n'_s}'^s) \in \Lambda_{n'_s}^\dagger$, $(s = \downarrow, \uparrow)$. The reason is as follows. If either $l(\nu^\downarrow - \delta'^\downarrow) > n_\downarrow$ or $l(\nu^\uparrow - \delta'^\uparrow) > n_\uparrow$ holds for the composition $\nu = (\nu^\downarrow, \nu^\uparrow) \in \Lambda_{n'}^{\downarrow\uparrow}$, then, in each monomial of the polynomials K_ν, the minimum power of $x_{n_\downarrow+1}$ is 1 or that of $x_{n'_\downarrow+n_\uparrow+1}$ is 1. Therefore, those ν do not contribute to the sum in (38) when both $x_{n_\downarrow+1}$ and $x_{n'_\downarrow+n_\uparrow+1}$ are set to be zero.

For a composition $\nu = (\nu^\downarrow, \nu^\uparrow) \in \Lambda_{n'}^{\downarrow\uparrow}$ satisfying $l(\nu^\downarrow - \delta'^\downarrow) \leqslant n_\downarrow$ and $l(\nu^\uparrow - \delta'^\uparrow) \leqslant n_\uparrow$, the composition $\mu = (\mu^\downarrow, \mu^\uparrow) \in \Lambda_n^{\downarrow\uparrow}$ can be defined as the composition satisfying the relation

$$\mu^\downarrow - \delta^\downarrow = \nu^\downarrow - \delta'^\downarrow \qquad \text{and} \qquad \mu^\uparrow - \delta^\uparrow = \nu^\uparrow - \delta'^\uparrow. \tag{39}$$

When relation (39) holds for compositions $\mu \in \Lambda_n^{\downarrow\uparrow}$ and $\nu \in \Lambda_{n'}^{\downarrow\uparrow}$, we can then find the following consequences. First, the two polynomials

$$\tilde{K}_\nu(x_1, \ldots, x_{n_\downarrow}, \underbrace{0, \ldots, 0}_{n'_\downarrow - n_\downarrow}, x_{n'_\downarrow+1}, \ldots, x_{n'_\downarrow+n_\uparrow}, \underbrace{0, \ldots, 0}_{n'_\uparrow - n_\uparrow}) \tag{40}$$

and

$$\tilde{K}_\mu(x_1, \ldots, x_{n_\downarrow}, x_{n'_\downarrow+1}, \ldots, x_{n'_\downarrow+n_\uparrow}) \tag{41}$$

are equal. From now on, we replace $x_{n'_\downarrow+1}, \ldots, x_{n'_\downarrow+n_\uparrow}$ by $x_{n_\downarrow+1}, \ldots, x_n$. Second, $\pi_\nu/d'_\nu = \beta^{|\nu|-|\mu|}\pi_\mu/d'_\mu$. Third, the expression $e_\nu/e_{\delta'}$ can be rewritten as

$$\frac{\prod_{s\in\nu}((a'(s)+1)/\beta+n'-l'(s))}{\prod_{s\in\delta'}((a'(s)+1)/\beta+n'-l'(s))} = \frac{\prod_{s\in\mu}((a'(s)+n'_\downarrow-n_\downarrow+1)/\beta+n'-l'(s))}{\prod_{s\in\delta}((a'(s)+n'_\downarrow-n_\downarrow+1)/\beta+n'-l'(s))}$$
$$= \beta^{|\delta|-|\mu|}\frac{(1+n'\beta+n'_\downarrow-n_\downarrow)_{\mu^+}}{(1+n'\beta+n'_\downarrow-n_\downarrow)_{\delta^+}}. \tag{42}$$

(Notice the relation $\prod_{s\in\eta}((a'(s)+k)/\beta+k'-l'(s)) = \beta^{-|\eta|}(k'\beta+k)_{\eta^\dagger}$ for integers j, k and $\eta\in\Lambda_n$.) As a result of these three relations, expression (38) can be rewritten as

$$\sum_\mu \beta^{-|\mu|}\frac{(1+n'\beta+n'_\downarrow-n_\downarrow)_{\mu^+}}{(1+n'\beta+n'_\downarrow-n_\downarrow)_{\delta^+}}\frac{\pi_\mu}{d'_\mu}\tilde{K}_\mu(x) \tag{43}$$

where the sum is taken over $\mu\in\Lambda_n^{\downarrow\uparrow}$.

Now we obtain the relation

$$\prod_{i\in I_\downarrow}(1-x_i)^{-n'\beta-n'_\downarrow}\prod_{j\in I_\uparrow}(1-x_j)^{-n'\beta-n'_\downarrow+n_\downarrow-n_\uparrow} = \sum_\mu \beta^{-|\mu|}\frac{(1+n'\beta+n'_\downarrow-n_\downarrow)_{\mu^+}}{(1+n'\beta+n'_\downarrow-n_\downarrow)_{\delta^+}}\frac{\pi_\mu}{d'_\mu}\tilde{K}_\mu(x) \tag{44}$$

from expressions (37) and (43). We notice that the right-hand side of (44) is a polynomial of $n'\beta+n'_\downarrow$ and hence relation (44) also holds for arbitrary complex values of $n'\beta+n'_\downarrow$. Further we replace $-n'\beta-n'_\downarrow+n_\downarrow$ by a complex variable a. Consequently we obtain binomial formula (33).

3. Hole propagator for the spin CS model

3.1. Result for a finite number of particles

In this section, for arbitrary non-negative integer β, we compute exactly the hole propagator of the SU(2) spin CS model with a finite number of particles.

We consider the $2M$-particle system whose Hamiltonian is given by (1) with $p=2$. We assume that M is an odd integer. The hole propagator of the model is given by

$$G(r,t) = {}_{2M}\langle 0|\hat{\psi}^\dagger_\uparrow(r,t)\hat{\psi}_\uparrow(0,0)|0\rangle_{2M}/{}_{2M}\langle 0|0\rangle_{2M} \tag{45}$$

where $|0\rangle_{2M}$ represents the singlet ground state for $2M$-particle system. The operator $\hat{\psi}_\uparrow(r,t) = \exp(\mathrm{i}\hat{H}_{2M-1}t)\hat{\psi}_\uparrow(r)\exp(-\mathrm{i}\hat{H}_{2M}t)$ is the Heisenberg representation of the annihilation operator $\hat{\psi}_\uparrow(r)$ of particles with spin-up which acts on the $2M$-particle states.

In the following, the statistics of particles are chosen as boson (fermion) for odd (even) β, so that we can set $P_{ij} = (-1)^{\beta+1}s_{ij}$. (Notice that $P_{ij}s_{ij}$ is nothing but the particle exchange operator for a pair (i, j).)

Along the lines of the calculations in [13], the hole propagator reduces to the following expression:

$$G(r,t) = c_0\prod_{i=1}^{2M-1}\oint_{|x_i|=1}\frac{\mathrm{d}x_i}{2\pi\mathrm{i}x_i}\bar{\Delta}^\beta(x)\bar{\Theta}(x;\beta)\exp[-\mathrm{i}(\tilde{\mathcal{H}}-E^0_{2M})t+\mathrm{i}\tilde{\mathcal{P}}r]\Delta^\beta(x)\Theta(x;\beta) \tag{46}$$

where complex coordinates $x = (x_1, \ldots, x_{2M-1})$ are related to original coordinates X in (1) by the formulae $x_i = \exp(2\pi \mathrm{i} X_i/L)$ for $1 \leqslant i \leqslant 2M-1$. The constant factor c_0 is given by $\rho_0/(2D(M))$, in terms of the mean density of particles $\rho_0 = 2M/L$ and

$$D(M) = \prod_{i=1}^{2M} \oint_{|z_i|=1} \frac{\mathrm{d}z_i}{2\pi \mathrm{i} z_i} \prod_{1\leqslant i<j\leqslant 2M} |z_i - z_j|^{2\beta} \prod_{1\leqslant i<j\leqslant M} |z_i - z_j|^2 \prod_{M+1\leqslant i<j\leqslant 2M} |z_i - z_j|^2$$
$$= \frac{M!}{(2\beta+1)^M} \frac{\Gamma((2\beta+1)M+1)}{\Gamma(\beta+1)^{2M}}. \tag{47}$$

In (46), the function $\Theta(x;\beta)$ has the form

$$\Theta(x;\beta) = \prod_{i=1}^{M} (1-x_i)^\beta \prod_{i=M+1}^{2M-1} (1-x_i)^{\beta+1} \prod_{1\leqslant i<j\leqslant M} (x_i - x_j) \prod_{M+1\leqslant i<j\leqslant 2M-1} (x_i - x_j). \tag{48}$$

Furthermore, the symbols $\tilde{\mathcal{H}}$, E^0_{2M} and $\tilde{\mathcal{P}}$ denote the Hamiltonian, the ground-state energy of $2M$-particle system and total momentum, respectively. In terms of the complex variables x, the expressions for $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{P}}$ are given by

$$\tilde{\mathcal{H}} = \left(\frac{2\pi}{L}\right)^2 \left[\sum_{i=1}^{2M-1} \left(x_i \frac{\partial}{\partial x_i} - \Delta P \right)^2 - \sum_{1\leqslant i<j\leqslant 2M-1} \frac{2\beta(\beta - (-1)^\beta s_{ij}) x_i x_j}{(x_i - x_j)^2} \right] \tag{49}$$

$$\tilde{\mathcal{P}} = \frac{2\pi}{L} \sum_{i=1}^{2M-1} \left(x_i \frac{\partial}{\partial x_i} - \Delta P \right) \tag{50}$$

respectively. Here ΔP denotes $(\beta(2M-1) + M - 1)/2$.

Now we introduce a transformed Hamiltonian $\hat{\mathcal{H}}$ and momentum $\hat{\mathcal{P}}$ as

$$\hat{\mathcal{H}} = \Delta^{-\beta} \tilde{\mathcal{H}} \Delta^\beta - E^0_{2M} \tag{51}$$

$$\hat{\mathcal{P}} = \Delta^{-\beta} \tilde{\mathcal{P}} \Delta^\beta. \tag{52}$$

Using the notations in section 2 with $n = 2M-1$, $n_\downarrow = M$, and $n_\uparrow = M-1$, expression (46) turns into

$$G(r,t) = c_0 \langle \Theta, \exp(-\mathrm{i}\hat{\mathcal{H}} t + \mathrm{i}\hat{\mathcal{P}} r) \Theta \rangle^0_{2M-1}. \tag{53}$$

In (53), our problem has reduced to the spectral decomposition of $\Theta(x;\beta)$ in terms of the joint eigenfunctions of $\hat{\mathcal{H}}$ and $\hat{\mathcal{P}}$. From the following two observations, we can see that the Jack polynomials with the prescribed symmetry are proper bases of the decomposition. First, both $\Theta(x)$ and $K_\mu(x)$ are polynomials with a common symmetric property; $s_i\Theta(x) = -\Theta(x)$ and $s_i K_\mu(x) = -K_\mu(x)$ for $i \in I_\downarrow \backslash \{n_\downarrow\}$ or $\in I_\uparrow \backslash \{n\}$. Second, the polynomials K_μ are joint eigenfunctions of $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{P}}$ with the eigenvalues

$$\omega(\mu) = \left(\frac{2\pi}{L}\right)^2 \sum_{i=1}^{2M-1} \left(\bar{\mu}_i - \frac{M-1}{2} + \beta \right)^2 \tag{54}$$

$$q(\mu) = \frac{2\pi}{L} \sum_{i=1}^{2M-1} \left(\mu_i - \frac{\beta(2M-1) + M - 1}{2} \right) \tag{55}$$

respectively. In the following, we discuss the second issue.

From expression (17), we can see that $E_\eta(x)$ for $\eta \in \Lambda_n$ and $x = (x_1, \ldots, x_n)$ is an eigenfunction of the operators $\sum_{i=1}^n \hat{d}_i^k$ with the eigenvalue $\sum_{i=1}^n \bar{\eta}_i^k$ for $k = 1, \ldots, n$. Furthermore, the operators $\sum_{i=1}^n \hat{d}_i^k$ for $k = 1, \ldots, n$ commute with permutations $\sigma \in S_{n_\downarrow} \times S_{n_\uparrow}$. In addition, K_μ can be constructed by a 'symmetrization' of E_μ (see expression

(22)). Therefore, the Jack polynomials with the prescribed symmetry K_μ for $\mu \in \Lambda_n^{\downarrow\uparrow}$ are eigenfunctions of $\sum_{i=1}^n \hat{d}_i^k$ with the eigenvalue $\sum_{i=1}^n \bar{\mu}_i^k$ for $k = 1, \ldots, n$.

On the other hand, in terms of the Cherednik–Dunkl operators (15), the Hamiltonian $\hat{\mathcal{H}}$ and total momentum $\hat{\mathcal{P}}$ can be written as

$$\hat{\mathcal{H}} = \left(\frac{2\pi}{L}\right)^2 \sum_{i=1}^{2M-1} \left(\hat{d}_i - \frac{M-1}{2} + \beta\right)^2 \tag{56}$$

and

$$\hat{\mathcal{P}} = \frac{2\pi}{L}\left[\sum_{i=1}^{2M-1}\left(\hat{d}_i - \frac{M-1}{2}\right) + \frac{\beta(2M-1)}{2}\right]. \tag{57}$$

These expressions lead to the fact that the joint eigenfunctions of $\sum_{i=1}^{2M-1} \hat{d}_i^k$ for $k = 1, 2$ are those of $\hat{\mathcal{H}}$ and $\hat{\mathcal{P}}$. From this, we find that the polynomials K_μ are joint eigenfunctions of $\hat{\mathcal{H}}$ and $\hat{\mathcal{P}}$. Eigenvalues (54) and (55) follow from (17), (56) and (57).

We notice that the binomial formula (33) is useful in rewriting (53), because the spectral decomposition of $\Theta(x;\beta)$ in terms of $K_\mu(x)$ is a special case of formula (33). Now we can express Θ in terms of K_μ as

$$\Theta(x;\beta) = \sum_{\mu\in\Lambda_{2M-1}^{\downarrow\uparrow}} \chi_\mu(\beta+M) K_\mu(x;\beta) \tag{58}$$

with the coefficients $\chi_\mu(\beta+M)$ (34).

Using the orthogonal properties of the Jack polynomials with the prescribed symmetry and the spectral decomposition (58), we have

$$G(r,t) = \frac{\rho_0}{2} \sum_{\mu\in\Lambda_{2M-1}^{\downarrow\uparrow}} (\chi_\mu(\beta+M))^2 \left(\frac{||K_\mu||_{2M-1}^0}{||K_{\delta(M,M)}||_{2M}^0}\right)^2 \exp[-\mathrm{i}(\omega(\mu)t - q(\mu)r)]. \tag{59}$$

(Notice that $K_{\delta(M,M)}(x) = \Delta(x^\downarrow)\Delta(x^\uparrow)$ and then $D(M) = (||K_{\delta(M,M)}||_{2M}^0)^2$.) Every factor in (59) is available from expressions (29), (34), (54) and (55). Expression (59) is our main result in this section.

It is important to consider the condition that intermediate states μ can contribute to the sum in (59). By a close examination of expression (34), we find the relation

$$\chi_\mu(\beta+M) \neq 0 \Leftrightarrow (1, \beta+M) \notin \mu^+. \tag{60}$$

This relation leads to a selection rule; only those states μ the largest entry of which is less than $\beta + M$ contribute to the sum in (59). Here we remark on the parametrization of the relevant intermediate state. For a composition $\mu \in \Lambda_n^{\downarrow\uparrow}$, there is a unique composition $\hat{\mu} = (\hat{\mu}^\downarrow, \hat{\mu}^\uparrow) \in \Lambda_n$ such that $\mu = \hat{\mu} + \delta$. Notice that $\hat{\mu}^s \in \Lambda_{n_s}^\dagger$ for $s = \downarrow, \uparrow$. Using these notations, we have

$$\chi_\mu(\beta+M) \neq 0 \Leftrightarrow (1, \beta+1) \notin \hat{\mu}^\downarrow \qquad \text{and} \qquad (1, \beta+2) \notin \hat{\mu}^\uparrow \tag{61}$$

where $\hat{\mu}^\downarrow$ and $\hat{\mu}^\uparrow$ are regarded as partitions. In the decomposition $\mu = \hat{\mu} + \delta$, δ represents the condensate or the pseudo-Fermi sea and $\hat{\mu}$ represents the excitations. From the above conditions, we see that partitions $\hat{\mu}^\downarrow$ and $\hat{\mu}^\uparrow$ are parametrized by β and $\beta + 1$ integers, respectively. Therefore, the relevant intermediate states μ can be parametrized by $2\beta + 1$ integers.

An example of those states is shown in figure 1, where we take $\beta = 2$, $M = 7$, and $\mu = (2, 2, 1, 1, 0, 0, 0, 3, 3, 2, 2, 1, 0) + \delta(n_\downarrow = 7, n_\uparrow = 6)$. The open blocks correspond to $\delta(n_\downarrow = 7, n_\uparrow = 6)$ and the shaded ones represent the excitations.

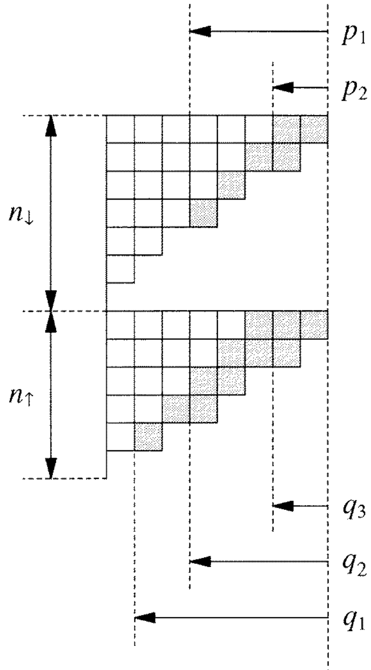

Figure 1. A figure that contributes to the hole propagator. This figure corresponds to the case with $\beta = 2$, $M = 7$ and $\mu = (2, 2, 1, 1, 0, 0, 0, 3, 3, 2, 2, 1, 0) + \delta(n_\downarrow = 7, n_\uparrow = 6)$. The open and shaded blocks represent the pseudo-Fermi sea and the excitation, respectively. The parameters $\{p_k, q_l\}$ adopted in section 3.2 are also shown.

In the next section, we will see that another but equivalent parametrization leads $(2\beta+1)$-fold integral representation for $G(r, t)$ in the thermodynamic limit.

3.2. Thermodynamic limit

In this section, we derive expression (2) for the hole propagator in the thermodynamic limit. First, to take the thermodynamic limit, we parametrize the intermediate states in the following way. Let $\mu = (\mu^\downarrow, \mu^\uparrow) \in \Lambda^{\downarrow\uparrow}_{2M-1}$ be a composition which satisfies condition (60). From the consideration on the intermediate states, the complementary set of $\{\mu_i^\downarrow\}_{i=1}^{M}$ in $\{M+\beta-1, M+\beta-2, \ldots, 1, 0\}$ is well defined. We define $\{p_k\}_{k=1}^{\beta}$ so that $p_1 > \cdots > p_\beta$ and $\{M+\beta-1, M+\beta-2, \ldots, 1, 0\}\backslash\{\mu_i^\downarrow\}_{i=1}^{M} = \{M+\beta-1-p_k\}_{k=1}^{\beta}$. Similarly, we define $\{q_l\}_{l=1}^{\beta+1}$ so that $q_1 > \cdots > q_{\beta+1}$ and $\{M+\beta-1, M+\beta-2, \ldots, 1, 0\}\backslash\{\mu_i^\uparrow\}_{i=1}^{M-1} = \{M+\beta-1-q_l\}_{l=1}^{\beta+1}$. The resultant set of $\{p_k, q_l\}$ exhausts the relevant intermediate state in (59). In figure 1, the new parameters $\{p_1, p_2, q_1, q_2, q_3\}$ are also shown.

Furthermore we shall introduce a set of notations. We respectively define $\tilde{p}_k (1 \leqslant k \leqslant \beta)$ and $\tilde{q}_l (1 \leqslant l \leqslant \beta + 1)$ by

$$\tilde{p}_k = p_k - \gamma\beta(\beta - k + \sharp\{l' \in \{1, \ldots, \beta + 1\} | q_{l'} < p_k\}) \tag{62}$$

$$\tilde{q}_l = q_l - \gamma\beta(\beta + 1 - l + \sharp\{k' \in \{1, \ldots, \beta\} | p_{k'} \leqslant q_l\}) \tag{63}$$

where $\gamma = 1/(2\beta + 1)$. We can regard $\tilde{p}_k$ and $\tilde{q}_l$ as 'rapidities' of quasiholes with spin-up and spin-down. Also, $\bar{\Delta}_k (1 \leqslant k \leqslant \beta)$ and $\hat{\Delta}_l (1 \leqslant l \leqslant \beta + 1)$ are defined as $\gamma\beta\sharp\{l' \in \{1, \ldots, \beta + 1\} | q_{l'} = p_k - 1\}$ and $\gamma\beta\sharp\{k' \in \{1, \ldots, \beta\} | p_{k'} = q_l - 1\}$, respectively. Moreover we introduce $U_{kl} = \tilde{p}_k - \tilde{p}_l - \bar{\Delta}_l$ for $1 \leqslant k < l \leqslant \beta$ and $V_{kl} = \tilde{q}_k - \tilde{q}_l - \hat{\Delta}_l$ for $1 \leqslant k < l \leqslant \beta + 1$. They describe the interplay between the quasiholes with the same spin. In order to describe the interaction between the quasiholes with an opposite spin, we introduce W_{kl} for $1 \leqslant k \leqslant \beta$ and $1 \leqslant l \leqslant \beta+1$ as

$$W_{kl} = \begin{cases} \tilde{p}_k - \tilde{q}_l - \gamma(\beta+1)\sharp\{k' \in \{1,\ldots,\beta\} | p_{k'} = q_l + 1\} & \text{for } q_l + 2 \leqslant p_k \\ 1 & \text{for } q_l \leqslant p_k \leqslant q_l + 1 \\ \tilde{q}_l - \tilde{p}_k + 1 - \gamma(\beta+1)\sharp\{l' \in \{1,\ldots,\beta+1\} | q_{l'} = p_k\} & \text{for } p_k \leqslant q_l - 1 \end{cases} \tag{64}$$

and

$$\tilde{\Delta}_{kl} = \begin{cases} \gamma\beta & \text{for } p_k = q_l \text{ or } p_k = q_l + 1 \\ 0 & \text{otherwise.} \end{cases} \tag{65}$$

Here $\tilde{\Delta}_{kl}$ is introduced to describe exceptional configurations of rapidities.

Using these new notations, we can rewrite the norm and matrix element more explicitly. The expression for $\chi_\mu(\beta+M)$ in the new notation is given by

$$\begin{aligned} \chi_\mu(\beta+M) = (-1)^{\sum_{k=1}^{\beta} p_k + \sum_{l=1}^{\beta+1} q_l - \beta^2} \prod_{k=1}^{2\beta+1} \Gamma(\gamma k)^{-1} \\ \times \prod_{k=1}^{\beta} \Gamma(\gamma(\beta+1) - \bar{\Delta}_k) \prod_{l=1}^{\beta+1} \Gamma(\gamma(\beta+1) - \hat{\Delta}_l) \\ \times \prod_{1\leqslant k<l\leqslant\beta} \frac{\Gamma(U_{kl} + \gamma(\beta+1))}{\Gamma(U_{kl})} \prod_{1\leqslant k<l\leqslant\beta+1} \frac{\Gamma(V_{kl} + \gamma(\beta+1))}{\Gamma(V_{kl})} \\ \times \prod_{k=1}^{\beta} \prod_{l=1}^{\beta+1} \frac{\Gamma(W_{kl} + \tilde{\Delta}_{kl} - \gamma\beta)}{\Gamma(W_{kl})}. \end{aligned} \tag{66}$$

The expression for the norm is given by

$$\begin{aligned} \left(\frac{||K_\mu||_{2M-1}^0}{||K_{\delta(M,M)}||_{2M}^0} \right)^2 = \frac{\Gamma(\beta+1)}{M} \frac{\Gamma(M(2\beta+1)-\beta)}{\Gamma(M(2\beta+1))} \prod_{k=1}^{2\beta+1} \frac{\Gamma(M+1-\gamma k)}{\Gamma(M+1-\gamma k - \gamma\beta)} \\ \times \prod_{k=1}^{\beta} \frac{\Gamma(1-\bar{\Delta}_k)\Gamma(\tilde{p}_k + \gamma(\beta+1))\Gamma(M - \tilde{p}_k - \gamma\beta)}{\Gamma(\gamma(\beta+1) - \bar{\Delta}_k)\Gamma(\tilde{p}_k + 1)\Gamma(M - \tilde{p}_k)} \\ \times \prod_{l=1}^{\beta+1} \frac{\Gamma(1-\hat{\Delta}_l)\Gamma(\tilde{q}_l + \gamma(\beta+1))\Gamma(M - \tilde{q}_l - \gamma\beta)}{\Gamma(\gamma(\beta+1) - \hat{\Delta}_l)\Gamma(\tilde{q}_l + 1)\Gamma(M - \tilde{q}_l)} \\ \times \prod_{1\leqslant k<l\leqslant\beta} \frac{\Gamma(U_{kl}+1)\Gamma(U_{kl})}{\Gamma(U_{kl} + \gamma(\beta+1))\Gamma(U_{kl} + \gamma\beta)} \\ \times \prod_{1\leqslant k<l\leqslant\beta+1} \frac{\Gamma(V_{kl}+1)\Gamma(V_{kl})}{\Gamma(V_{kl} + \gamma(\beta+1))\Gamma(V_{kl} + \gamma\beta)} \\ \times \prod_{k=1}^{\beta} \prod_{l=1}^{\beta+1} \frac{\Gamma(W_{kl})\Gamma(W_{kl} + \tilde{\Delta}_{kl})}{\Gamma(W_{kl} + \gamma\beta)\Gamma(W_{kl} + \tilde{\Delta}_{kl} - \gamma\beta)}. \end{aligned} \tag{67}$$

Now we consider the thermodynamic limit, i.e., $M \to \infty$, $L \to \infty$ with $\rho_0 = 2M/L$ fixed. In this limit, only the configurations with p_k, q_l, $|p_k - p_l|$, $|q_k - q_l|$ and $|p_k - q_l| \sim \mathcal{O}(M)$ give finite contributions to the hole propagator. Let us introduce the normalized velocities u_k and v_l of the holes with up- and down-spin, respectively. These are defined by

$$\lim_{M\to\infty} \frac{2p_k}{M} = 1 - u_k \tag{68}$$

$$\lim_{M\to\infty} \frac{2q_l}{M} = 1 - v_l. \tag{69}$$

In the thermodynamic limit, we can use the Stirling formula: $\Gamma(z+1) \to \sqrt{2\pi} z^{z+1/2} \exp(-z)$ for $|z| \to \infty$. Taking the symmetry of the integrand into consideration, the sum in (59) over $\{p_k, q_l\}$ reduces to the integral over $\{u_k, v_l\}$ as

$$\sum_{0\leqslant p_\beta<\cdots<p_1\leqslant M+\beta-1} \sum_{0\leqslant q_{\beta+1}<\cdots<q_1\leqslant M+\beta-1} \to \frac{1}{\beta!(\beta+1)!} \left(\frac{M}{2}\right)^{2\beta+1} \prod_{k=1}^{\beta} \int_{-1}^{1} \mathrm{d}u_k \prod_{l=1}^{\beta+1} \int_{-1}^{1} \mathrm{d}v_l. \tag{70}$$

Combining expressions (59), (66) and (67) with the above limiting procedure, we arrive at the final expression (2) for the hole propagator in the thermodynamic limit.

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